MATH 2028 - Applications of Fubini's Theorem

<u>GOAL</u>: Explore more applications of Fubini's Thm and evaluate some multiple integrals

We start with some simple examples.

<u>Example 1</u>: Evaluate $\int_R f\,dV$ where

$$f : R = [0,1] \times [1,2] \to \mathbb{R}$$

$$f(x,y) := 1 + x^2 + xy$$

<u>Solution</u>: First, we note that f is cts on R, hence integrable. Thus, Fubini's Theorem applies.

$$\int_R f\,dV = \int_0^1 \int_1^2 1 + x^2 + xy \, dy\, dx$$

$$= \int_0^1 \left[y + x^2 y + \frac{1}{2} x y^2 \right]_{y=1}^{y=2} dx$$

$$= \int_0^1 1 + x^2 + \frac{3}{2} x \, dx$$

$$= \left[x + \frac{1}{3} x^3 + \frac{3}{4} x^2 \right]_{x=0}^{x=1} = \frac{25}{12}$$

Alternatively, we can also do the iterated integral in the reversed order.

$$\int_R f \, dV = \int_1^2 \int_0^1 1 + x^2 + xy \, dx \, dy$$

$$= \int_1^2 \left[x + \frac{1}{3}x^3 + \frac{1}{2}x^2 y \right]_{x=0}^{x=1} dy$$

$$= \int_1^2 \frac{4}{3} + \frac{1}{2} y \, dy$$

$$= \left[\frac{4}{3} y + \frac{1}{4} y^2 \right]_{y=1}^{y=2} = \frac{25}{12}$$

$\square$

Sometimes it is easier to compute the iterated integrals in a particular order.

<u>Example 2</u> : Evaluate $\int_R f \, dV$ where

$$f : R = [0,2] \times [-1,1] \longrightarrow \mathbb{R}$$

$$f(x,y) := x y e^{x+y^2}$$

Solution: First, we note that f is cts on R, hence integrable. Thus, Fubini's Theorem applies.

$$\int_R f\,dV = \int_0^2 \int_{-1}^1 xye^{x+y^2}\,dy\,dx$$

$$= \int_0^2 xe^x \underbrace{\left(\int_{-1}^1 ye^{y^2}\,dy\right)}_{=0 \ \because \text{ odd function of } y} dx = 0$$

Doing it in a different order.

$$\int_R f\,dV = \int_{-1}^1 \int_0^2 xye^{x+y^2}\,dx\,dy$$

$$= \int_{-1}^1 ye^{y^2}\left(\int_0^2 xe^x\,dx\right) dy$$

$$= \int_{-1}^1 ye^{y^2}\left[xe^x - e^x\right]_{x=0}^{x=2} dy$$

$$= (e^2+1)\underbrace{\int_{-1}^1 ye^{y^2}\,dy}_{=0} = 0$$

$\blacksquare$

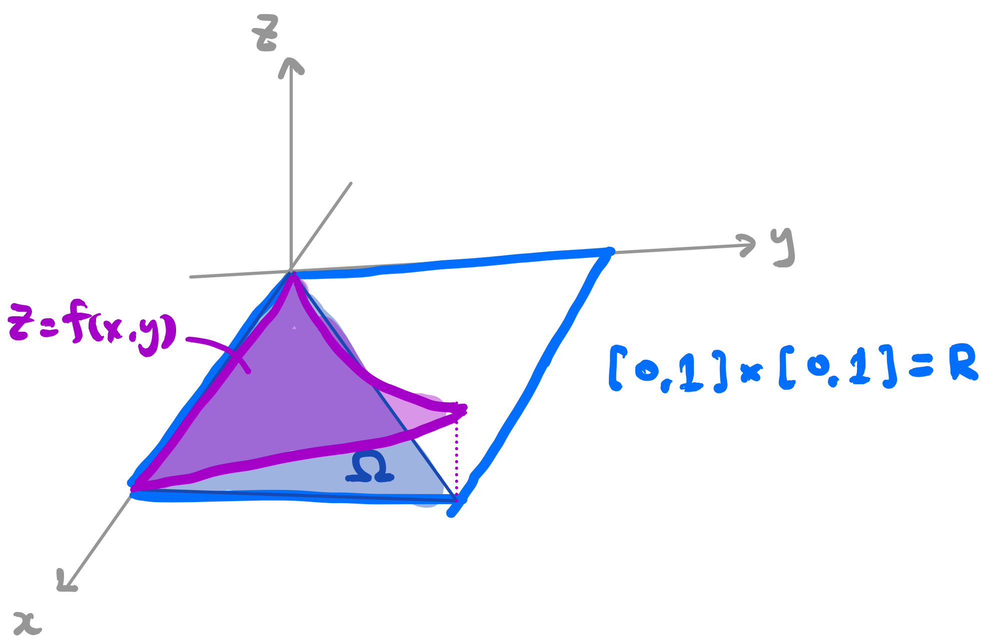
We can also use Fubini's Theorem to evaluate integrals on a non-rectangular bdd $\Omega \subseteq \mathbb{R}^n$.

<u>Example 3</u>: Find the volume of the region lying over the triangle

$$\Omega = \{ (x,y) \in [0,1] \times [0,1] \mid x \geq y \}$$

and below the graph of $f : \Omega \to \mathbb{R}$ defined by

$$f(x,y) = xy.$$



<u>Note</u>: Volume of the region $= \int_\Omega f \, dV$

<u>Solution</u>: Since f is cts on Ω and $\partial\Omega$ has measure zero, f is integrable on Ω and

the extension $\bar{f}$ is integrable on R. We can apply Fubini's Theorem to evaluate $\int_R \bar{f}\, dV$.

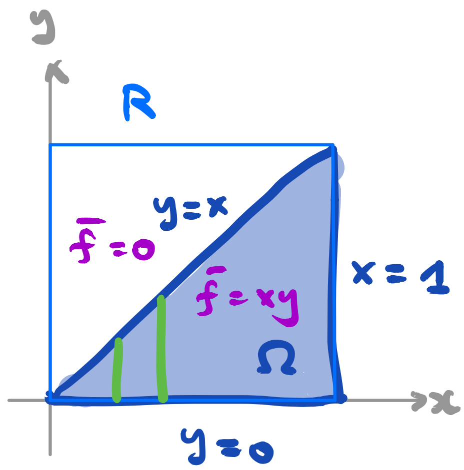
$$\int_\Omega f\, dV \overset{\text{by def}^{\text{n}}}{=} \int_R \bar{f}\, dV \overset{\text{Fubini}}{=} \int_0^1 \int_0^1 \bar{f}(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^x xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2} x y^2\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{1}{2} x^3\, dx$$

$$= \left[\frac{1}{8} x^4\right]_{x=0}^{x=1} = \frac{1}{8}$$



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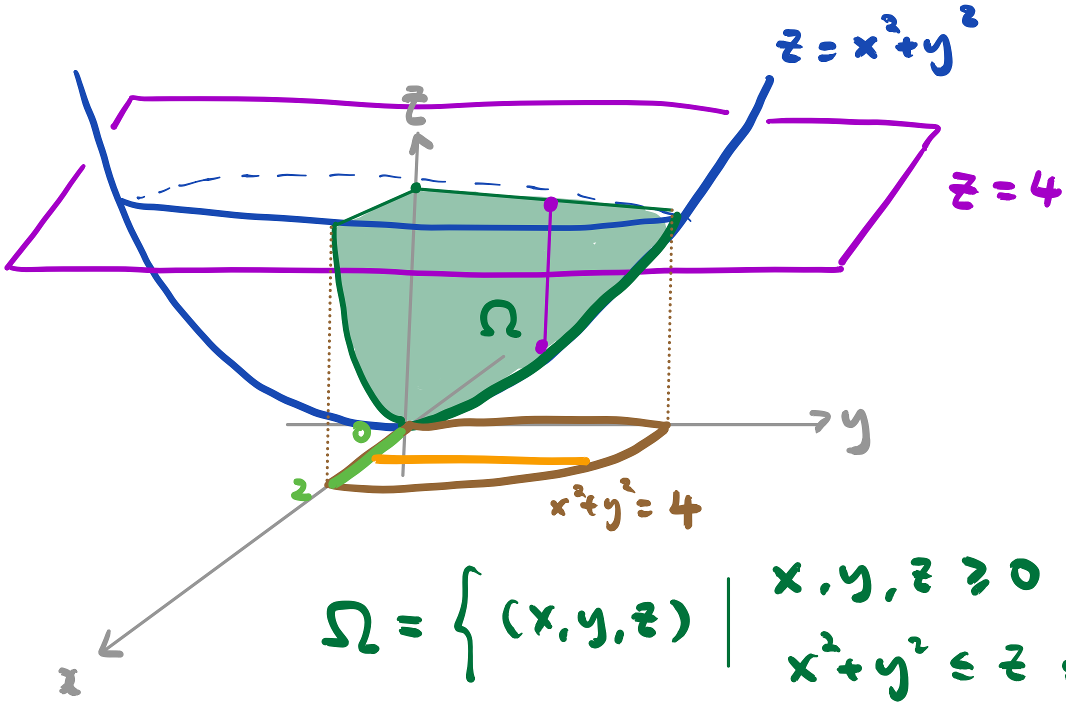
Example 4: Let Ω be the region in the 1st octant bdd below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Evaluate the integral

$$\int_\Omega x\, dV$$

Solution: **Step 1:** Visualize the region



$$\Omega = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x,y,z \geq 0 \quad \text{and} \\ x^2+y^2 \leq z \leq 4 \end{array} \right\}$$

Since $f(x,y,z) := x$ is cts on Ω and $\partial\Omega$ has measure zero. f is integrable on Ω.

Step 2: Setup the iterated integral and evaluate.

Choose the rectangle $R = [0,1] \times [0,1] \times [0,4]$ which enclose the region Ω.

$$\int_\Omega f \, dV = \int_R \bar{f} \, dV$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} x \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} x\,(4 - x^2 - y^2)\, dy\, dx$$

$$= \int_0^2 x \left[(4 - x^2)\, y - \frac{1}{3} y^3 \right]_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= \int_0^2 \frac{2}{3} x\,(4 - x^2)^{3/2}\, dx$$

$$= \left[-\frac{2}{15} (4 - x^2)^{5/2} \right]_{x=0}^{x=2} = \frac{64}{15}$$

$\blacksquare$

Let's see one example in n dimension.

Example 5: Find the volume of

$$\Omega = \{ (x_1, \dots, x_n) : 0 \leq x_n \leq x_{n-1} \leq \cdots \leq x_1 \leq 1 \}$$

Solution:

$$\mathrm{Vol}(\Omega) := \int_\Omega 1\, dV = \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-1}} dx_n \cdots dx_2\, dx_1$$

$$= \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-2}} x_{n-1}\, dx_{n-1} \cdots dx_2\, dx_1$$

$$= \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-3}} \frac{1}{2} x_{n-2}^2\, dx_{n-2} \cdots dx_2\, dx_1 = \frac{1}{n!}$$

$\blacksquare$

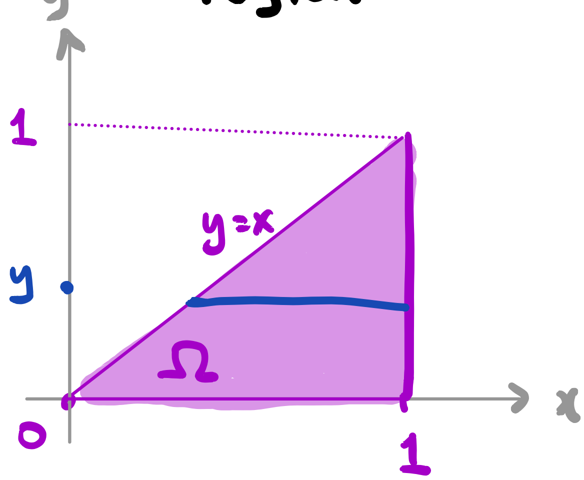
Sometimes we can turn Fubini's theorem around to help us evaluate certain iterated integrals.

Example 6: Evaluate the iterated integral

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$$

Solution: Note that $\int \frac{\sin x}{x}\, dx$ cannot be integrated in closed form. We want to flip the order of integration to see if it helps.

Step 1: Identify the region



Step 2: Define the function

$$f(x,y) = \begin{cases} \frac{\sin x}{x} & \text{if } x = 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is then a cts function on Ω.

Note: $\partial\Omega$ has measure zero.

Step 3: Apply Fubini's Theorem.

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy = \int_\Omega f\, dV$$

$$= \int_0^1 \int_0^x f(x,y)\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x} \cdot x\, dx = 1 - \cos 1$$

$\square$

Finally, we look at an example where both iterated integrals exist BuT f is NOT integrable.

Example 7: Consider $f: R = [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x = \frac{m}{q},\ y = \frac{n}{q} \text{ for some } m,n,q \in \mathbb{N},\ q \text{ prime} \\ 0, & \text{otherwise.} \end{cases}$$

Claim: f is NOT integrable on R

Pf of Claim: Since both $\mathbb{Q}^c$ and the set

$$\left\{ \frac{m}{q} \;\middle|\; m, q \in \mathbb{N},\ q \text{ prime} \right\}$$

are dense in $[0,1]$, we have $L(f, \mathcal{P}) = 0$ and $U(f, \mathcal{P}) = 1$ for <u>ANY</u> partition $\mathcal{P}$ of R.

Hence, f is NOT integrable.

Next, we compute the iterated integrals.

- when $x \neq \frac{m}{q}$, $\int_0^1 f(x,y)\,dy = 0$
- when $x = \frac{m}{q}$, also $\int_0^1 f(x,y)\,dy = 0$

  since $f(\frac{m}{q}, y) = 0$ except for finitely many y

Therefore, $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = 0$

Similarly, we also have $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = 0$

So, both iterated integrals exist and are equal to zero BuT f is <u>NOT</u> integrable.

$\blacksquare$